Lecture 3

CS 59000 Reinforcement Learning (RL)

Agenda:

- Integral.
- Expectation
- change of measures.
- Concentration of measures.

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measure space $(\Omega, \mathcal{F}, \mu)$

Ω → measurable space.
$\mathcal{F}$ → σ algebra on Ω
μ → measure

$A \in \mathcal{F}$ is called measurable set.

$\Omega = \{0, 1\}$ (0 → T, 1 → H) → $0 \rightarrow -2$, $1 \rightarrow 5$

$\mathcal{F} = \{\phi, \{0,1\}, \{0\}, \{1\}\}$

$\mathbb{P}(\phi) = 0$ $\quad \underline{\mathbb{P}(\{0,1\}) = 1}$, $\quad \mathbb{P}(\underline{\{0\}}) = 1/5$, $\mathbb{P}\{\{1\}\} = 4/5$

$(\Omega, \mathcal{F}, \mathbb{P})$

$X : \Omega \rightarrow R$

$X(0) \rightarrow -2$

$X(1) \rightarrow 5$

$X(\{0,1\}) \rightarrow \{-2, 5\}$

$\mathbb{P}_X(\{-2\}) = 1/5 \qquad \mathbb{P}_X(\{5\}) = 4/5$

$\Omega = \{(0,0), (0,1), (1,0), (1,1)\}$

$\mathcal{F}_0 = \{\emptyset, \Omega\}$

$\mathcal{F}_1 = \{\underbrace{\{(0,0),(0,1)\}}_{1/5}, \underbrace{\{(1,0),(1,1)\}}_{4/5}, \emptyset, \Omega\}$

$\mathcal{F}_2 = \{\{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}\} \cup \mathcal{F}_1$

$, \{(1,0),(0,0)\}, \{(0,1),(1,1)\}, \{(0,1)\} - - -)$

$\mathbb{P}((0,0)) = \frac{1}{100}$, $- - -$

Filtration. $\mathcal{F}_i \subseteq \mathcal{F}_{i+1}$ for all i

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$(\Omega, \mathcal{F}, \mathbb{P})$

Consider a measure space $(\Omega, \mathcal{F}, \mu)$. An indicator function $I_A(\omega)$ is 1 if $\omega \in A$, and zero o.w.

Ω 

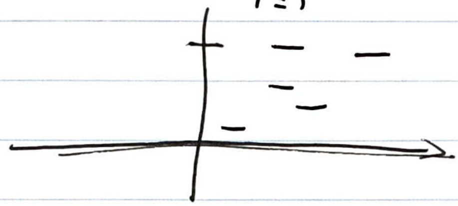
we define integral of $I_A(\omega)$ w.r.t the measure μ, as follows:

$$\int I_A \, d\mu = \int I_A(\omega) \, d\mu(\omega) = \mu(A)$$

for A in $\mathcal{F}$.

Def: A function h on a measurable space Ω whose rang consists of only finitly many points in $\mathbb{R}^+$ is called a "simple" funtion.

$$h = \sum_{i=1}^{n} \alpha_i \cdot I_{A_i}$$ where $\alpha_1, \dots, \alpha_n$ are distinct, and $A_i = \{\omega : h(\omega) = \alpha_i\}$



Remark: $\int h \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i)$

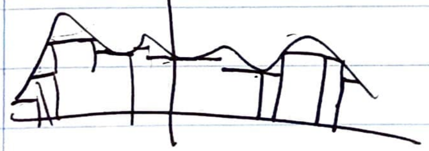
$$\int \left(\sum_{i=1}^{n} \alpha_i \cdot I_{A_i}\right) d\mu = \sum_{i=1}^{n} \int \alpha_i \cdot I_{A_i} \, d\mu$$

$$= \sum_{i=1}^{n} \alpha_i \underbrace{\int I_{A_i} \cdot d\mu} = \sum \alpha_i \mu(A_i)$$

Def: let $X : \Omega \to \mathbb{R}$ be a measurable funtion on $(\Omega, \mathcal{F}, \mu)$. then we have:

$$\int X \, d\mu = \int X(\omega) \, d\mu(\omega)$$

$$= \sup_h \left\{ \int h \, d\mu : h \text{ is simple } 0 \leq h \leq X \right\}$$

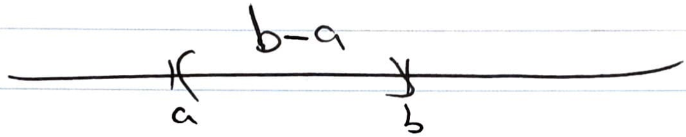


4)

If the sup is infinity, then the integral of X is not defined

Borel

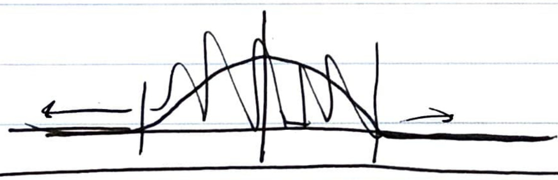
If $\Omega = R$, $\mathcal{F}$ ~~is called Lebesgue~~ σ-algebra
where the Lebesgue measure λ is unique
such that $\lambda((a,b)) = b-a$ for $a \leq b$



Def. Consider two measures, λ and μ on $(\Omega, \mathcal{F})$
λ is absolutely continuous w.r.t μ i.e.

$$\lambda \ll \mu$$

if $\lambda(A) = 0$ for every $A \in \mathcal{F}$
for which $\mu(A) = 0$



If $\lambda \ll \mu$, then there exist a funtion h
where $\int |h| \, d\mu < \infty$ ($h \in L^1(\mu)$), such that

$$\lambda(A) = \int_A h \, d\mu \quad \text{for } A \in \mathcal{F}$$

$$h = \frac{d\lambda}{d\mu} \qquad \Leftarrow \; = \int_A d\lambda$$

h is called Radon-Nikodym derivative.

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Note: when μ is a Lebesgue measure, we have heard the term density for h.

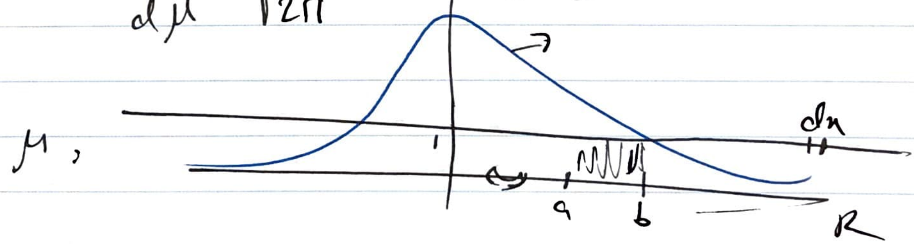
Example: Consider a standard Gaussian measure λ on $(\mathbb{R}, \mathcal{F})$ where $\mathcal{F}$ is Lebesgue σ-algebra

$$\lambda(A) = \frac{1}{\sqrt{2\pi}} \int_A \underbrace{\exp\left(-\frac{1}{2}x^2\right)} \underset{\downarrow d\mu}{dx}$$

$$\lambda(A) = \frac{1}{\sqrt{2\pi}} \int_A \underbrace{\exp\left(-\frac{1}{2}x^2\right)} \underline{d\mu}$$

$$\hookrightarrow \quad d\lambda = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) d\mu$$

$$\frac{d\lambda}{d\mu} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$



Decomposition: $\int X \, d\mu =$

$$= \int X \, I(X \geq 0) \, d\mu - \int (-X) \, I(X < 0) \, d\mu$$

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Def: Let $X: \Omega \to \mathbb{R}$ be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ then, the expectation of X is as follows:

$$E[X] = \int X \, d\mathbb{P} = \int x \, d\mathbb{P}_X$$

Conditional Expectation:

For simplicity, let's consider, discret and finite random varibles X am Z.

Lets consider the case that X can take values $x_1, \dots, x_m$ and Z take values $z_1, \dots, z_n$.

Now, note that: $\mathbb{P}(X = x_i | Z = z_j)$

$$= \frac{\mathbb{P}\{X = x_i \text{ and } Z = z_j\}}{\mathbb{P}(Z = z_j)}$$

provided that $\mathbb{P}(Z = z_j) > 0$.

Then, the conditional expectation is

$$E[X | Z = z_j] = \sum_{i=1}^{m} x_i \cdot \mathbb{P}(X = x_i | Z = z_j)$$

note that $E[X | Z = z_j]$ is a number:

Let's define $E[X|Z]$ as a random variable where $E[X|Z](\omega) = E[X | Z = z_j]$, in $\omega \in Z^{-1}(z_j)$

Here $E[X|Z]$ is a function.

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We also know

$$E[X] = \sum_{j=1}^{n} E[X|Z = z_j] \, P(Z = z_j)$$

We define $Y = E[X|Z]$ a random variable

which take values $y_1, \ldots, y_n$ such that

$$y_j = E[X|Z = z_j]. \rightarrow \text{range}$$

note that $Y(\omega) = y_j$ for $\omega \in Y^{-1}(y_j)$
which means $Y(\omega)$ is constant for all $\omega \in Y^{-1}(y_j)$
In other words, you do not need to know which ω is the input. The only thing you need is what is $Y^{-1}(y_j)$.. which is a set.

If we look at Ω, and all y_j, then $Y^{-1}(y_j)$ partitions Ω

for all $\omega \rightarrow Y(\omega) = y_j$

. Let call A_j, the set that $Z(\omega) = z_j$
i.e. $A_j = \{\omega : Z(\omega) = z\}$.

Construct G, as the smalles σ-algebra that

$$A_j \in G \quad \forall j \in [n]$$

8)

Then, Y is $\mathcal{G}$-measurable.

It means. $Y^{-1}(A) \in \mathcal{G}$ for all, Borel $A \subset \mathbb{R}$

Note that, since $Y(\omega)$ is constant y_j when $\omega \in A_j$

where $E[Y 1_{A_j}] = \int_{A_j} Y \, dP$

constant y_j

$$= y_j \int_{A_j} dP = y_j \mathbb{P}(A_j)$$

$\mathbb{P}(Z = z_j)$

$$= \sum_{i=1}^{m} x_i \, \mathbb{P}(X = x_i \mid Z = z_j) \, P(Z = z_j)$$

$$= \sum_{i=1}^{m} x_i \, \mathbb{P}(X = x_i, Z = z_j)$$

$$= \int_{A_j} X \, d\mathbb{P}$$

$$\rightarrow \quad \int_A Y \, d\mathbb{P} = \int_A X \, dP \quad \forall A \in \mathcal{G}$$

9)

Consider $\mathcal{G}$ to be a sub-σ-algebra of $\mathcal{F}$.
For a random variable X on $(\Omega, \mathcal{F}, \mu)$, with $E[|X|]<\infty$ ($X \in L^1(\mu)$) there exist another random variable Y, $\mathcal{G}$-measurable, and $E[|Y|]<\infty$ such that for all $G \in \mathcal{G}$,

$$\int_G \underset{E[X|\mathcal{G}](G)}{Y} \, dP = \int_G X \, dP$$

Y is called a version of the conditional expectation and we write $Y = E[X|\mathcal{G}]$, a.s.

10)

Some properties:

- If $X \geq 0$, then $E[X|G] \geqslant 0$ a.s.
- $E[1|G] = 1$ a.s. almost surely.
- $E[X+Y|G] = E[X|G] + E[Y|G]$ a.s.
- $E[XY|G] = Y E[X|G]$ a.s. if $E[XY]$ exist, and Y is G-measurable.
- If $G_1 \subset G_2$, then $E[X|G_1] = E[E[X|G_2]|G_1]$
- If $G = \{\phi, \Omega\}$ is the trivial σ-algebra a.s.

$$E[X|G] = E[X] \quad \text{a.s.}$$

11 :)

traditional way

$X \rightarrow \quad E[X \mid X<5]$

$\Omega = \{1,2,3,4,5,6\} \rightarrow \mathcal{F} = 2^6$

$\mathcal{G} = \{\{1,2,3\}, \{4,5,6\}, \phi, \Omega\}$

$X = I \qquad X(5) = 5$

$E[X] = \frac{21}{6}$

$E[X \mid X>3] = \frac{1}{3} \times 6 + \frac{1}{3} \times 5 + \frac{1}{3} \times 4$

$Y = E[X \mid \mathcal{G}](\{1,2,3\}) = \frac{1}{3} \times 1 + \frac{1}{3} \times 2 + \frac{1}{3} \times 3$

$E[X \mid \mathcal{G}](\{4,5,6\}) = \frac{1}{3} 6 + \frac{1}{3} \times 5 + \frac{1}{3} \times 4,$